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Norm Estimates for the Inverses of a General Class of Scattered-Data Radial-Function Interpolation Matrices

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In this paper we investigate those radial basis functions h associated with functions whose mth derivative (modulo a scalar multiple) is completely monotonic. Our results apply both to interpolation problems that require polynomial reproduction and to those that do not. In the case where polynomial reproduction is not required and the order m is 0 or 1, we obtain estimates on the norms of inverses of scattered-data interpolation matrices. These estimates depend only on the minimal-separation distance for the data and on the dimension of the ambient space, $\mathbb{R}^s$. When the order m satisfies $m \geq 2$, we show that there exist parameters $a_1, ..., a_m$ such that the function $h(x)+a_m+a_{m-1}r^2+ \cdots +a_1 r^{2m-2}$ gives rise to an invertible interpolation matrix, and we obtain bounds on the norm of the inverse of this matrix. For interpolation methods in which one wishes to reproduce polynomials of total degree $m-1$ or less, bounds for the norm of the inverse of the interpolation matrix are obtained, provided the data contains a $\pi_{m-1}(\mathbb{R}^s)$ unisolvent subset. These results apply, in particular, to Duchon's "thin-plate spline" results. © 1992 Academic Press, Inc.

I. INTRODUCTION

Over the last few years, there has been considerable progress concerning the theoretical development of data fitting in two or more dimensions. In particular, the method of thin-plate splines, as developed by Duchon [2], and, more recently, the results of Micchelli [9] and Madych and Nelson [7, 8] concerning radial basis functions are notable examples. In each of these papers, certain classes of interpolation matrices associated with scattered data in $\mathbb{R}^s$ were shown to be invertible. These results established the "well-poisedness" of the scattered data interpolation problem relative to the families of either the thin-plate splines or certain radial basis functions, such as the Hardy multiquadrics [6].

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More recently, progress has been made in "quantifying" these interpolation methods, in the sense of estimating the norms of the inverses of these interpolation matrices as well as their condition numbers. For example, in [1] such estimates were determined for matrices associated with the function $h(x) = \|x\|_2$. In [10], a general approach, using Fourier transform techniques, was developed for making such estimates for interpolation matrices coming from conditionally negative definite radial (CNDR) functions of order one. Previous to these papers, there were some useful numerical results on condition numbers obtained by Dyn, Levin, and Rippa [3].

In this paper, we employ the methods introduced in [10] to estimate certain quadratic forms associated with the family of Gaussians, $e^{-r^2 t}$, $t > 0$. We first use these estimates to get an upper bound on $\|A^{-1}\|$ for interpolation matrices A determined by CNDR functions that are of order zero or one and that are generated by completely monotonic functions with corresponding orders. The estimates for $\|A^{-1}\|$ are given at the end of Section II. In addition to the dependence on the CNDR interpolation function, the estimates for $\|A^{-1}\|$ depend only on the minimal separation distance for the data set and the dimension s of the ambient space, $\mathbb{R}^s$. For interpolation methods employing CNDR functions that are of order $m \geqslant 2$ and which are associated with functions whose mth derivative is completely monotonic, we have results for two different interpolation problems.

These results greatly extend the results given in [10] in that the results obtained in this paper apply to a much broader class of functions and they hold for arbitrary dimension.

Let h be a CNDR function generated by a completely monotonic function of order $m \geqslant 2$. For interpolation methods in which one wishes to reproduce polynomials of total degree $m-1$ or less and where the interpolating function has the form

$$\sum_{j=1}^{N} c_j h(x - x_j) + \sum_{|\alpha| < m} k_\alpha x^\alpha, \qquad \text{with} \quad \sum_{j=1}^{N} c_j x_j^\alpha = 0, \quad |\alpha| < m,$$

bounds for the norm of the inverse of the interpolation matrix are obtained, provided the data contains a $\pi_{m-1}(\mathbb{R}^s)$ unisolvent subset. These results are given in Section IV. In particular, they are relevant to Duchon's "thin-plate spline" interpolation [2].

In Section V we deal with the problem in which h is a CNDR function of order $m \geqslant 1$ and where polynomial reproduction is not required. We show that there exist parameters $a_1, ..., a_m$ such that the shifts to the interpolation points of the function

$$h(x) + a_m + a_{m-1} r^2 + \cdots + a_1 r^{2m-2}$$

give rise to an invertible interpolation matrix, and we obtain bounds on the norm of the inverse of this matrix. We also show that if slightly worsened bounds are tolerated, the parameters may be chosen so that the resulting interpolation matrix is negative definite. We mention that the parameters and bounds appear to depend on the details of the distribution of the data when $m \geqslant 2$.

In the closing section of our paper we apply our results to certain radial basis functions and to thin-plate splines. We now turn to a precise discussion of the interpolation problems mentioned above.

Background. Throughout the remainder of this paper, we will be considering two types of interpolation problems that we now proceed to describe.

Given a continuous function $h\colon \mathbb{R}^s \to \mathbb{R}$, vectors $\{x_j\}_1^N$ in $\mathbb{R}^s$, and scalars $\{y_j\}_1^N$, one version of the scattered data interpolation problem consists of finding a function f such that the system of equations

$$f(x_j) = y_j, \qquad j = 1, ..., N$$

has a solution of the form

$$f(x) = \sum_{j=1}^{N} c_j h(x - x_j). \tag{1.1}$$

Equivalently, one wishes to know when the $N \times N$ matrix A with entries $A_{j,k} = h(x_j - x_k)$ is invertible.

In the second version of the scattered data interpolation problem, the interpolant is required to have the form

$$s(x) = \sum_{j=1}^{N} c_j h(x - x_j) + \sum_{|\alpha| < m} k_\alpha x^\alpha, \tag{1.2}$$

where the constants c_j and k_α must satisfy

$$\sum_{j=1}^{N} c_j h(x_i - x_j) + \sum_{|\alpha| < m} k_\alpha x_i^\alpha = y_i, \qquad i = 1, ..., N \tag{1.2a}$$

$$\sum_{j=1}^{N} c_j x_j^\alpha = 0, \qquad |\alpha| < m. \tag{1.2b}$$

This second method guarantees polynomial reproduction in case the data contains a $\Pi_{m-1}(\mathbb{R}^s)$ unisolvent subset.

The following class of functions has played a prominent role in the study of both scattered-data problems [3–5, 7–12].

DEFINITION 1.1. *Let $h: \mathbb{R}^s \to \mathbb{C}$ be continuous. We say that h is conditionally negative definite of order m if for every finite set $\{x_j\}_1^N$ of distinct points in $\mathbb{R}^s$ and for every set of complex scalars $\{c_j\}_1^N$ satisfying*

$$\sum_{j=1}^{N} c_j q(x_j) = 0, \qquad \forall q \in \pi_{m-1}(\mathbb{R}^s),$$

we have $\sum_1^N \bar{c}_j c_k h(x_k - x_j) \leqslant 0$. We denote by $\mathcal{N}_m^s$ the class of these functions.

In what follows, we will assume that $\mathbb{R}^s$ is endowed with a norm $\|\cdot\|$. We define the function $v: \mathbb{R}^s \to \mathbb{R}^+$ by $v(x) = \|x\|$.

DEFINITION 1.2. *We will say that a continuous function $F: [0, \infty) \to \mathbb{R}$ is a conditionally negative definite radial function of order m if $F \circ v$ is in $\mathcal{N}_m^s$. We will denote the set of all such functions by $\mathcal{RN}_m^s$. For the special case $m = 0$, we say that $G \in \mathcal{RP}_0^s$ if $F := -G$ is in $\mathcal{RN}_0^s$.*

Note that if the norm used is $\|\cdot\|_2$, the Hilbert space norm, and if $s_1 \leqslant s_2 \leqslant \infty$, then one has the inclusions $\mathcal{RN}_m^{s_1} \supset \mathcal{RN}_m^{s_2} \supset \mathcal{RN}_m^{\infty}$. The class $\mathcal{RN}_m^{\infty}$ includes those functions F which are continuous on $[0, \infty)$ and for which $(-1)^{m+1}(d^m/d\sigma^m)F(\sqrt{\sigma})$ is completely monotonic on $(0, \infty)$ [16]; we will denote the class of such F by $\mathcal{RN}_{m,c}^{\infty}$. The inclusion $\mathcal{RN}_m^{\infty} \supset \mathcal{RN}_{m,c}^{\infty}$ is due to Schoenberg [12] in the $m = 0$ case and to Micchelli [9] in the case where $m \geqslant 1$. The reverse inclusion is also known for $m = 0$ (Schoenberg [12]) and for $m = 1$ (Micchelli [9]). For $m > 1$, the reverse inclusion is apparently true due to a recent result of K. Guo, S. Hu, and X. Sun.

When h is conditionally negative definite of order zero, $-h$ is a positive definite function (in the sense of Bochner), and the matrix $-A$ is nonnegative definite. It will be invertible if and only if the quadratic form Q associated with $-A$ is a positive definite. Moreover, $\|A^{-1}\|$ is precisely the reciprocal of the minimum of Q over all unit vectors. For the case of CNDR functions that are of order one, there is again a connection between Q and $\|A^{-1}\|$.

In what follows all norm symbols will refer to the l_2 norm.

LEMMA 1.3 (K. Ball [1]). *Let $\{x_j\}_1^N$ be distinct points in $\mathbb{R}^s$ and let $F \in \mathcal{RN}_1^s$ be nonnegative and suppose that $h(x) = F(\|x\|)$ is a strictly conditionally negative definite function of order 1. Also, let A be the matrix with entries $A_{j,k} = h(x_j - x_k)$. If the inequality*

$$\sum_{j,k=1}^{N} A_{jk} \xi_j \bar{\xi}_k \leqslant -\theta \sum_{j=1}^{n} |\xi_j|^2$$

is satisfied whenever the complex numbers ξ_j *satisfy* $\sum_{j=1}^N \xi_j = 0$, *then*

$$\|A^{-1}\| \leqslant \theta^{-1}.$$

The proof of this result involves elementary matrix theory and will be omitted. The result itself is important because it is useful for obtaining estimates on $\|A^{-1}\|$ in the $m=1$ case and because it suggests the connection between norm estimates and quadratic form estimates in cases where $m \geqslant 2$, a connection that we exploit in obtaining our results.

We close this section with a result that will be of use later.

LEMMA 1.4 [9]. *Let* $k = 1, 2, 3, \ldots$. *If* $\sum_1^N c_i p(x_i) = 0$ *for all* $p \in \pi_{k-1}(\mathbb{R}^s)$, *then*

$$(-1)^k \sum_1^N \sum_1^N c_i c_j \|x_i - x_j\|^{2k} \geqslant 0, \tag{1.3}$$

where equality holds in (1.3) *if and only if*

$$\sum_1^N c_i p(x_i) = 0, \qquad p \in \pi_k(\mathbb{R}^s). \tag{1.4}$$

II. BASIC ESTIMATES

In this section we derive an upper estimate on certain quadratic forms. In addition, we obtain estimates on $\|A^{-1}\|$ in the $m=0$ and $m=1$ cases. The quadratic form estimates will also be used later, in connection with the various interpolation problems mentioned in Section I. We begin with the following lemma, which is found in [9]. For completeness, we include a short proof here.

LEMMA 2.1. *Let* $g\colon [0, \infty) \to \mathbb{R}$ *be continuous on* $[0, \infty)$ *and let* $\varepsilon > 0$ *be arbitrary. If* $(-1)^m g^{(m)}$ *is completely monotonic on* $(0, \infty)$, *then on* $[0, \infty)$ *there exists a nonnegative Borel measure* $d\eta(t)$ *for which*

$$g(\sigma) = \sum_{j=0}^{m-1} \frac{g^{(j)}(\varepsilon)(\sigma-\varepsilon)^j}{j!} + \int_0^\infty \frac{1}{t^m} \left\{ e^{-\sigma t} - \left(\sum_{j=0}^{m-1} \frac{(-1)^j (\sigma-\varepsilon)^j}{j!} t^j \right) e^{-\varepsilon t} \right\} d\eta(t).$$

Proof. Since $(-1)^m g^{(m)}$ is completely monotone, there exists [16] a unique nonnegative Borel measure $d\eta(t)$ such that

$$(-1)^m g^{(m)}(\sigma) = \int_0^\infty e^{-t\sigma}\, d\eta(t). \tag{2.1}$$

Next, let σ be an arbitrary point in $(0, \infty)$. Using Taylor's Theorem with remainder about the point $\sigma = \varepsilon$, we have

$$g(\sigma) = \sum_{j=0}^{m-1} \frac{g^{(j)}(\varepsilon)}{j!}(\sigma - \varepsilon)^j + \frac{1}{(m-1)!}\int_\varepsilon^\sigma g^{(m)}(\sigma')(\sigma' - \sigma)^{m-1}\, d\sigma'.$$

Substituting (2.1) into the integral expression above yields the desired result. ∎

In what follows, V_k, $k = 0, 1, 2, \ldots$, will be the subspace

$$V_k := \{v \in \mathbb{R}^N : v_j = p(x_j),\ p \in \pi_k(\mathbb{R}^s),\ j = 1, \ldots, N\} \tag{2.2}$$

while $V_k^\perp$ will be the orthogonal complement of V_k relative to $\mathbb{R}^N$.

COROLLARY 2.2. *Let $F \in \mathscr{RN}^\infty_{m,c}$, and let Q be the associated quadratic form below. Then, provided $\xi = (\xi_i, \ldots, \xi_N) \in V_{m-1}^\perp$, it follows that*

$$\begin{aligned} Q &:= -\sum_{j,k=1}^N F(\|x_j - x_k\|)\xi_j\xi_k \\ &= \int_0^\infty \frac{Q_t}{t^m}\, d\eta(t), \end{aligned} \tag{2.3}$$

where Q_t is the quadratic form given by

$$Q_t := \sum_{j,k=1}^N \xi_j \xi_k e^{-\|x_j - x_k\|^2 t}. \tag{2.4}$$

Proof. Note that $F(r) = -g(r^2)$ for some function g for which $(-1)^m g^{(m)}$ is completely monotonic. Note also that by Lemma 1.4, the quadratic form involving the polynomial part of F is zero and hence our claim is validated. ∎

We wish to obtain a lower bound on Q restricted to the subspace $V_{m-1}^\perp$. Corollary 2.2 allows us to obtain such a bound by first getting explicit, positive, lower bounds on the quadratic form Q_t defined in (2.4), then substituting them into (2.3), and finally integrating. To describe the lower bound on Q_t, we need to introduce some notation. As in [10], we let q be half the smallest distance between any two points in our data set

$\{x_1, ..., x_N\}$; we will call q the *separation radius.* We assume that the data set comprises N distinct points, so that $q > 0$. The lower bound on Q_t can now be stated.

THEOREM 2.3. *For every $\xi \in \mathbb{R}^N$ and every $t \geq 0$, we have that*

$$Q_t \geq \phi(t) \, \|\xi\|^2, \qquad \text{where} \quad \phi(t) := C_s t^{-s/2} q^{-s} e^{-\delta^2 q^{-2} t^{-1}}, \tag{2.5}$$

where C_s and δ are constants given by

$$\delta := 12 \left(\frac{\pi \Gamma^2((s+2)/2)}{9} \right)^{1/(s+1)} \qquad \text{and} \qquad C_s := \frac{\delta^2}{2^{s+1} \Gamma((s+2)/2)}.$$

The proof of this theorem is technical and is deferred to Section III. Using the lower estimate for Q_t stated above, we easily obtain the following lower estimate on Q.

THEOREM 2.4. *Let $F \in \mathscr{RN}^{\infty}_{m,c}$, let $\xi = \{\xi_j\}_{j=1}^N \in V_{m-1}^{\perp}$, and set*

$$\theta := \int_0^{\infty} \frac{\phi(t)}{t^m} \, d\eta(t), \tag{2.6}$$

where $\phi(t)$ appears in Theorem 2.3 and $d\eta(t)$ is the measure for the completely monotonic function $(-1)^m g^{(m)}(t)$, $g(t)$ being the function $-F(\sqrt{t})$. Then, provided $g^{(m)}$ is nonconstant, the following holds:

$$\sum_{i,j=1}^{N} F(\|x_i - x_j\|) \xi_i \xi_j \leq -\theta \, \|\xi\|^2.$$

Proof. By Theorem 2.3, it follows that for each fixed t,

$$\sum_1^N \xi_i \xi_j e^{-\|x_i - x_j\|^2 t} \geq \phi(t) \, \|\xi\|^2.$$

Substitution of this inequality in (2.3) yields the result. ∎

In case $m = 0$ or $m = 1$, we can use this theorem to obtain the following norm-estimates on inverses of the interpolation matrices associated with radial functions in $\mathscr{RN}^{\infty}_{m,c}$.

COROLLARY 2.5. *Let $F \in \mathscr{RN}^{\infty}_{0,c}$ be nonconstant. If θ is given by (2.6) with $m = 0$, then, for any finite subset $\{x_1, ..., x_N\} \in \mathbb{R}^s$, with N and s arbitrary positive integers, the $N \times N$ interpolation matrix A, with entries $A_{jk} = F(\|x_j - x_k\|)$, has an inverse that satisfies*

$$\|A^{-1}\| \leq \theta^{-1}.$$

Proof. F being nonconstant implies that the corresponding completely monotonic function $g(\sigma) := -F(\sqrt{\sigma})$ is generated by a measure $d\eta$ that is supported on some Borel set not containing $\sigma = 0$. Hence, the integral defining θ in (2.6) is strictly positive, which in turn implies that $-A$ is positive definite and that θ is a lower bound on the smallest eigenvalue of $-A$. The norm estimate then follows from elementary matrix theory. ∎

COROLLARY 2.6. *Let $F \in \mathcal{RN}^{\infty}_{1,c} = \mathcal{RN}^{\infty}_{1}$. Suppose that $r^{-1}F' = 2g'(r^2)$ is non-constant on $(0, \infty)$ and that $F(0) \geqslant 0$. If θ is given by (2.6) with $m = 1$, then, for any finite subset $\{x_1, ..., x_N\} \in \mathbb{R}^s$, with N and s arbitrary positive integers, the $N \times N$ interpolation matrix A, with entries $A_{jk} = F(\|x_j - x_k\|)$, has an inverse that satisfies*

$$\|A^{-1}\| \leqslant \theta^{-1}.$$

Proof. The fact that $r^{-1}F' = 2g'(r^2)$ is non-constant guarantees that the measure $d\eta$ appearing in (2.1) is supported on a Borel subset of $(0, \infty)$. Consequently, θ in (2.6) (with $m = 1$) is again positive. Since $F(0) \geqslant 0$ implies that $F \in \mathcal{RN}^{\infty}_{1}$ is nonnegative, an application of Lemma 1.3 yields the result. ∎

III. A LOWER BOUND FOR THE QUADRATIC FORM Q_t

The quadratic form Q_t defined in (2.4) is associated with the Gaussian function

$$F_t(r) := e^{-r^2 t}, \tag{3.1}$$

where $t > 0$. Our aim is to find a positive lower bound for

$$\theta(t) := \min_{\xi \neq 0 \in \mathbb{R}^s} \frac{Q_t}{\sum_{j=1}^{N} \xi_j^2}.$$

Indeed, we will show that $\theta(t)$ has the lower bound given in Theorem 2.3. To find this lower bound, we will use an adaptation of the method introduced in [10, Sect. IV].

At first glance, it appears that the techniques developed in [10] for estimating lower bounds on quadratic forms such as Q_t cannot be directly employed. The lower estimates there were obtained at the expense of constraining the components of $\xi \in \mathbb{R}^s$; they had to satisfy $\sum_{j=1}^{N} \xi_j = 0$. Fortunately, this constraint was used only to get a certain integral representation for the quadratic form being estimated. For the function $F_t(r)$, this integral representation can be obtained without employing the constraint. Consequently, the results from [10] *can* be used to obtain lower estimates for Q_t.

To obtain the appropriate integral representation for Q_t, first write $F_t(r) = e^{-r^2 t}$ in terms of its well-known Fourier transform over $\mathbb{R}^s$ [14, p. 9]. (The variable r is, for this purpose, regarded as the Euclidean norm in $\mathbb{R}^s$.) Once this is done, replace the integral over the angle variables by the function Ω_s [14, p. 26],

$$\Omega_s(u \|x\|) = \omega_{s-1}^{-1} \int_{S_{s-1}} e^{iu\langle x, \eta\rangle} \, d\sigma_{s-1}(\eta), \tag{3.2}$$

where S_{s-1}, ω_{s-1}, and $d\sigma_{s-1}$ are, respectively, the unit sphere in $\mathbb{R}^s$, its volume, and the usual measure on it. (For future reference, we note that Ω_s and ω_{s-1} have the following explicit formulas [14, pp. 26–27]:

$$\Omega_s(z) = \Gamma(s/2)(2/z)^{(s-2)/2} J_{(s-2)/2}(z) \tag{3.3}$$

and

$$\omega_{s-1} = \frac{2\pi^{s/2}}{\Gamma(s/2)}. \tag{3.4}$$

Here, $\Gamma(\cdot)$ denotes the Gamma function and $J_p(\cdot)$ denotes the order p Bessel function of the first kind.) The result of this replacement is the radial representation

$$e^{-r^2 t} = \int_0^\infty \Omega_s(ur) \, d\gamma_t(u), \tag{3.5}$$

where $d\gamma_t(u) = \omega_{s-1}(4\pi t)^{-s/2} e^{-u^2/4t} u^{s-1} \, du$. Finally, insert (3.5) into (2.4) and interchange the finite sum and the integral; this gives the following representation for Q_t.

LEMMA 3.1. *The quadratic form Q_t given in* (2.4) *has the form*

$$Q_t = \int_0^\infty \left[\sum_{j,k=1}^{N} \Omega_s(u \|x_j - x_k\|) \xi_j \xi_k \right] \frac{d\alpha_t(u)}{u^2}, \tag{3.6}$$

with $d\alpha_t$ given by

$$d\alpha_t(u) = \omega_{s-1}(4\pi)^{-s/2} t^{-s/2} e^{-u^2/4t} u^{s+1} \, du. \tag{3.7}$$

This is the integral representation that we are seeking. The minimization procedure from [10, Sect. IV] may be directly applied to it, without the imposition of any constraint. Also, to avoid confusion, be aware that this representation arises in connection with F_t being a positive definite radial function; it is *not* directly connected with the integral representations discussed in Section II.

According to the method introduced in [10, Sect. IV], finding the lower estimate $\theta(t)$, as given by [10, (4.12)], begins with finding a function $\chi\colon \mathbb{R}^s \to \mathbb{R}$ having Fourier transform $\hat{\chi}$ that satisfies

(i) $\hat{\chi} > 0$

(ii) $\hat{\chi}$ is a radial function and

(iii) $d\alpha_t(u)/u^2 \geqslant \hat{\chi}(u)\, u^{s-1}\, du$.

Then, $\theta(t)$ satisfies the inequality $\theta(t) \geqslant c_1(\chi(0) - c_2) > 0$. The precise constants c_1 and c_2 are given in (3.15).

There are infinitely many functions that satisfy these criteria. Selecting a useful χ hinges on the convergence of a certain series that depends on χ. (See [10, Sect. IV, (4.12)].) We will produce a family of such functions, and select the one that best serves our purposes. To do that, we need to look at the function $\hat{\psi}_\beta$, the characteristic function of the $\mathbb{R}^s$-ball centered at the origin and having radius β.

This function is radial and has a radial (inverse) Fourier transform given by

$$\psi_\beta(x) = (2\pi)^{-s} \int_{\mathbb{R}^s} e^{-i\langle x, \xi\rangle} \hat{\psi}_\beta(\xi)\, d^s\xi.$$

Integrating over the angle variables and using (3.2) yield

$$\psi_\beta(x) = \omega_{s-1}(2\pi)^{-s} \int_0^\beta \Omega_s(u\, \|x\|)\, u^{s-1}\, du. \tag{3.8}$$

Substituting (3.3) and (3.4) into (3.8) and changing the variable of integration form u to $v = \|x\|\, u$ result in

$$\psi_\beta(x) = (2\pi)^{-s/2}\, \|x\|^{-s} \int_0^{\|x\|\,\beta} J_{(s/2)-1}(v)\, v^{s/2}\, dv.$$

The integral on the right above can be evaluated explicitly [15, p. 360]; the result is

$$\psi_\beta(x) = \left(\frac{\beta}{2\pi\, \|x\|}\right)^{s/2} J_{s/2}(\|x\|\, \beta). \tag{3.9}$$

Using (3.9) and the series expansion for J_p found in [15, p. 359], one easily sees that ψ_β is analytic in $\|x\| \neq 0$ and that it has a removable singularity at $\|x\| = 0$, with

$$\psi_\beta(0) = \frac{\beta^s}{(4\pi)^{s/2}\Gamma((s+2)/2)} > 0. \tag{3.10}$$

The function that wil be used for χ will be a multiple of ψ_β^2. This will ensure that (i) wil be satisfied, since $\hat{\chi}$ will be the convolution of two positive functions. Let us therefore define the function

$$\chi(x) = K\psi_\beta^2(x) = K\left(\frac{\beta}{2\pi\,\|x\|}\right)^s J_{s/2}^2(\|x\|\,\beta). \tag{3.11}$$

We will select β later. To determine K, observe that the Convolution Theorem implies that

$$\hat{\chi}(\xi) = (2\pi)^{-s} K\hat{\psi}_\beta * \hat{\psi}_\beta(\xi).$$

Since $\hat{\psi}$ is the characteristic function for the ball of radius β, center 0, the convolution product above is nonnegative and has its support contained in a ball centered at 0 and having radius 2β. Also, because it is the Fourier transform of a radial function, $K\psi_\beta^2(x)$, it is itself radial. Thus $\hat{\chi}$ satisfies two of the three criteria imposed on it. The last of the three,

$$\frac{d\alpha_t(u)}{u^2} \geqslant \hat{\chi}(u)u^{s-1}\,du$$

will determine K. (Writing $\hat{\chi}(u)$ is an abuse of notation. It should not cause any difficulty, though, for $\hat{\chi}(\xi)$ is radial and therefore is constant for $\|\xi\| = u = \text{constant}$. With identical justification, we will also write $\chi(r)$.)

From the definition of $\hat{\chi}$ and standard theorems concerning the convolution, one has that

$$\hat{\chi}(\xi) \leqslant (2\pi)^{-s} K\,\|\hat{\psi}_\beta\|_2^2.$$

The square of the L^2-norm of $\hat{\psi}_\beta$ is just the volume of the sphere in $\mathbb{R}^s$ with radius β and center 0, and so (3.4) and the last equation yield

$$\begin{aligned}\hat{\chi}(\xi) &\leqslant (2\pi)^{-s} K s^{-1}\beta^s\omega_{s-1}\\ &= \frac{2\beta^s}{s(4\pi)^{s/2}\Gamma(s/2)}K.\end{aligned}$$

From this inequality, (3.7), and the support of $\hat{\chi}$ being contained in the ball with center 0 and radius 2β, it is clear that the third criterion will be satisfied by choosing

$$K = s\left(\frac{\pi}{t\beta^2}\right)^{s/2} e^{-\beta^2/t}. \tag{3.12}$$

Combining (3.11) and (3.12) yields the following result.

LEMMA 3.2. *Let $\beta > 0$. A function $\chi\colon \mathbb{R}^s \to \mathbb{R}$ having Fourier transform $\hat{\chi}$ that satisfies the crieria* (i), (ii), *and* (iii) *is given by*

$$\chi(x) = s(4t\pi\,\|x\|^2)^{-s/2} e^{-\beta^2/t} J^2_{s/2}(\|x\|\,\beta). \tag{3.13}$$

Having found a family of appropriate χ's, we may now proceed with the next step in the procedure. From [10, Sect. IV], Q_t's lower bound $\theta(t)$ may be estimated as follows.

First, recall that in Section II we defined the separation radius q of the data as half the minimum distance between two data points. Second, let

$$\kappa_n := \sup\{|\chi(x)| : nq \leqslant \|x\| \leqslant (n+1)q\}. \tag{3.14}$$

The estimate on $\theta(t)$, as given in [10] and described in the discussion following (3.7), is then given by

$$\theta(t) \geqslant \frac{(2\pi)^s}{\omega_{s-1}}(\chi(0) - 3^s\Sigma), \qquad \text{where} \quad \Sigma = \sum_{n=1}^{\infty} n^{s-1}\kappa_n. \tag{3.15}$$

From (3.10), (3.11), and (3.12), we have

$$\chi(0) = K\psi^2_\beta(0) = s\left(\frac{\beta^2}{16\pi t}\right)^{s/2} e^{-\beta^2/t}\left(\Gamma\left(\frac{s+2}{2}\right)\right)^{-2} \tag{3.16}$$

The κ_n's are somewhat harder to estimate in a useful way. To do so requires an inequality involving Bessel functions; this inequality is stated below.

LEMMA 3.3. *For $s = 1, 2, \ldots$, and for all $z > 0$, $J^2_{s/2}(z) \leqslant 2^{s+2}/z\pi$.*

Proof. When $s = 1$, from [13, p. 297] we have $J^2_{s/2}(z) \leqslant 2/z\pi < 2^{1+2}/z\pi$. When $s \geqslant 2$, Weber's "crude" inequality [13, p. 211] for $H^{(1)}_{s/2}(z)$, the order $s/2$ Hankel function, implies that

$$|H^{(1)}_{s/2}(z)|^2 \leqslant \frac{2}{z\pi}\left(1 - \frac{s-1}{4z}\right)^{-s-1} \qquad \text{with} \quad 4z > s-1.$$

For z real, $J_{s/2}(z) = \Re\{H_{s/2}^{(1)}(z)\}$. Thus from the last inequality we get

$$J_{s/2}^2(z) \leqslant \frac{2}{z\pi}\left(1 - \frac{s-1}{4z}\right)^{-s-1} \qquad \text{with} \quad 4z > s-1. \tag{3.17}$$

Now, it is easy to show that the maximum of $zJ_{s/2}^2(z)$ must occur for z greater than the first positive root of $J'_{s/2}$. One can estimate this root from below by $\sqrt{(s/2)(s+4)/2} > (s-1)/2$. (See [13, p. 487].) This fact and (3.17) imply that, *for all* $z > 0$,

$$zJ_{s/2}^2(z) \leqslant \frac{2}{\pi}(1-(1/2))^{-s-1} = \frac{2^{s+2}}{\pi}, \tag{3.18}$$

which immediately yields the desired inequality. ∎

We can use the inequality in the lemma to estimate $|\chi(x)|$. From Lemma 3.3 and (3.13), we obtain the inequality

$$|\chi(x)| \leqslant \frac{4s}{\beta} t^{-s/2} \pi^{-1-s/2} \|x\|^{-1-s} e^{-\beta^2/t},$$

and, consequently, that

$$\begin{aligned}\kappa_n &:= \sup\{|\chi(x)| : nq \leqslant \|x\| \leqslant (n+1)q\} \\ &\leqslant \frac{4s}{\beta} t^{-s/2} \pi^{-1-s/2} (nq)^{-1-s} e^{-\beta^2/t}.\end{aligned} \tag{3.19}$$

Our next goal is to use the estimate for κ_n in (3.19) to estimate from above the sum Σ appearing in (3.15). From the expression given for Σ in (3.15) and the inequality in (3.19), it follows that

$$\Sigma = \sum_{n=1}^{\infty} n^{s-1}\kappa_n \leqslant \frac{4s}{\beta} t^{-s/2}\pi^{-1-s/2} q^{-1-s} e^{-\beta^2/t} \sum_{n=1}^{\infty} n^{-2}.$$

Using the well-known formula $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$, we arrive at

$$\Sigma \leqslant \frac{2s\pi}{3\beta}(t\pi)^{-s/2} q^{-1-s} e^{-\beta^2/t}.$$

From this inequality, (3.15), and (3.16), it follows that

$$\begin{aligned}\theta(t) &\geqslant \frac{(2\pi)^s}{\omega_{s-1}}(\chi(0) - 3^s\Sigma) \\ &\geqslant \frac{(2\pi)^s}{\omega_{s-1}}\chi(0)\left[1 - \frac{\pi\Gamma^2((s+2)/2)}{18}(\beta q/12)^{-s-1}\right].\end{aligned}$$

By using (3.4), (3.16), and the identity $\Gamma(p+1)=p\Gamma(p)$, we may transform the lower estimate for $\theta(t)$ given above into the form

$$\theta(t) \geqslant \frac{\beta^s}{2^s t^{s/2} \Gamma((s+2)/2)} e^{-\beta^2/t} \left[1 - \frac{\pi \Gamma^2((s+2)/2)}{18} (\beta q/12)^{-s-1}\right]. \tag{3.20}$$

Recall that $\beta > 0$ was left unspecified. Let us choose it to satisfy

$$\left[1 - \frac{\pi \Gamma^2((s+2)/2)}{18} (\beta q/12)^{-s-1}\right] = 1/2.$$

Solving this equation yields

$$\beta := \delta/q, \qquad \text{where} \quad \delta := 12 \left(\frac{\pi \Gamma^2((s+2)/2)}{9}\right)^{1/(s+1)}. \tag{3.21}$$

Inserting (3.21) into (3.20) results in our final lower estimate for $\theta(t)$:

$$\theta(t) \geqslant C_s t^{-s/2} q^{-s} e^{-\delta^2 q^{-2} t^{-1}}, \qquad \text{where} \quad C_s := \frac{\delta^s}{2^{s+1} \Gamma((s+2)/2)}. \tag{3.22}$$

As a corollary to the construction used to obtain the estimate on $\theta(t)$, we have these results.

THEOREM 3.4. *Suppose that $F \in \mathscr{R}\mathscr{N}_0^s$ has the representation*

$$F(r) = \int_0^\infty \frac{\Omega_s(ur)}{u^2}\, d\alpha(u), \qquad d\alpha(u) = u^{s+1} \rho(u)\, du,$$

where $\rho(u)$ is strictly positive, decreasing, and continuous on $(0, \infty)$. With δ as given in (3.21), one has

$$\|A^{-1}\| \leqslant \frac{2sq^s}{\delta^s \rho(2\delta/q)}.$$

Proof. Follow the construction of χ up to the point where the constant K appearing in (3.11) is determined. Note that in this case the argument used to get (3.12) results in

$$\frac{2\beta^s}{s(4\pi)^{s/2} \Gamma(s/2)} K \leqslant \rho(2\beta),$$

so that one may choose

$$K = \frac{s(4\pi)^{s/2} \Gamma(s/2)}{2\beta^s} \rho(2\beta) \qquad \text{and} \qquad \chi(x) = \frac{s\rho(2\beta)}{2(\pi \|x\|^2)^s} J_{s/2}^2(\|x\|\, \beta).$$

To obtain an estimate from below for the θ that is the minimum for the quadratic form Q associated with F, one may now use steps identical to the ones leading up to the estimate (3.22) to arrive at

$$\theta \geqslant \frac{\delta^s \rho(2\delta/q)}{2sq^s},$$

from which the desired norm estimate can be obtained as an immediate consequence of elementary matrix theory. ∎

Similar reasoning used in conjunction with Lemma 1.3 and the estimates from [10, Sect. IV] yield a similar result for the $m=1$ case:

THEOREM 3.5. *Suppose that* $F \in \mathscr{RN}_1^s$ *has the representation*

$$F(r) = F(0) + \int_0^\infty \frac{1-\Omega_s(ur)}{u^2}\, d\alpha(u), \qquad d\alpha(u) = u^{s+1}\rho(u)\, du,$$

where $\rho(u)$ *is strictly positive, decreasing, and continuous on* $(0, \infty)$ *and* $F(0) \geqslant 0$. *With* δ *as given in* (3.21), *one has*

$$\|A^{-1}\| \leqslant \frac{2sq^s}{\delta^s \rho(2\delta/q)}.$$

IV. INTERPOLATION WITH POLYNOMIAL REPRODUCTION

In this section, we apply the results of the previous section to obtain invertibility criteria for interpolation matrices arising from functions $F \in \mathscr{RN}_{m,c}^\infty$, in the case of interpolation with polynomial reproduction.

Consider the interpolation problem as described by (1.2), (1.2a), and (1.2b). In matrix notation the interpolation problem converts to finding, for given $y \in \mathbb{R}^N$, vectors $c \in \mathbb{R}^N$ and $k \in \mathbb{R}^{m'}$ such that

$$\begin{aligned} Ac + Bk &= y \\ B^T c &= 0, \end{aligned} \tag{4.1}$$

where A is an $N \times N$ matrix and B is $N \times m'$ with $m' = \dim \pi_{m-1}(\mathbb{R}^s)$. We further assume that the data set $\{x_i\}_1^N$ contains a $\pi_{m-1}(\mathbb{R}^s)$ unisolvent set so that the matrix B has rank m'. In this case, the system (4.1) can be reduced to the single matrix equation

$$Gz := AP^\perp z + Pz, \qquad G\colon \mathbb{R}^N \to \mathbb{R}^N,$$

where P denotes the orthogonal projection onto the space V_{m-1} defined in (2.2).

We will now proceed both to show that G is invertible and to obtain bounds on $\|G^{-1}\|$. Recall that if U and W are subspaces of $\mathbb{R}^N$, then the angle $\alpha \in [0, \pi/2]$ between them is defined to be that angle for which $\cos \alpha$ is largest and

$$|\langle u, w \rangle| \leqslant \cos \alpha \, \|u\| \, \|w\| \tag{4.2}$$

holds when $u \in U$ and $w \in W$. Using compactness and duality, one may show that there exists $u_0 \in U$ such that $\|u_0\| = 1$ and such that

$$\sin \alpha = \sup\{|\langle u_0, w' \rangle| : w' \in W^{\perp} \text{ and } \|w'\| = 1\}. \tag{4.3}$$

This formula yields the following result:

LEMMA 4.1. *Let α be the angle between $U = \text{Range}(AP^{\perp})$ and $W = \text{Range } P = V$. Then, it follows that*

$$\sin \alpha \geqslant \frac{\theta}{A}, \tag{4.4}$$

where θ is as in Theorem 2.4 *and $A = \|AP^{\perp}\|$.*

Proof. From our assumption, the vectors u_0 and w' appearing in (4.3) satisfy

$$\begin{cases} u_0 = AP^{\perp} z_0 & \text{and} \qquad w' = P^{\perp} z \\ \|u_0\| = 1 & \text{and} \qquad \|P^{\perp} z\| = 1. \end{cases}$$

Since $\|u_0\| = 1$, $u_0 \neq 0$ and so $P^{\perp} z_0 \neq 0$. From (4.3), we then have

$$\sin \alpha \geqslant \frac{|\langle AP^{\perp} z_0, P^{\perp} z_0 \rangle|}{\|P^{\perp} z_0\|}.$$

In this inequality, we can remove the restriction on the norm of $AP^{\perp} z_0$ by simply dividing and multiplying by an appropriate factor. Doing this and also using the self-adjointness of $P^{\perp}$ yields

$$\sin \alpha \geqslant \frac{|\langle P^{\perp} A P^{\perp} z_0, z_0 \rangle|}{\|z_0\|^2} \cdot \frac{\|z_0\|^2}{\|AP^{\perp} z_0\| \cdot \|P^{\perp} z_0\|}.$$

The inequality (4.4) then follows from the last inequality and Theorem 2.4. ∎

The estimate on $\sin\alpha$ in Lemma 4.1 and the bound on the quadratic form appearing in Theorem 2.4 suffice to prove the invertibility of G and to estimate $\|G^{-1}\|$.

THEOREM 4.2. *The matrix G is invertible and satisfies*

$$\|G^{-1}\| \leqslant \frac{\sqrt{2}\,\Lambda}{\theta} \max\{1, \theta^{-1}\}, \tag{4.5}$$

where θ is as in Theorem 2.4 and $\Lambda = \|AP^{\perp}\|$.

Proof. Let U and W be as in Lemma 4.1. In addition, let $u = AP^{\perp}z$ and $w = Pz$, where $z \in \mathbb{R}^N$; clearly, $Gz = u + w$. From the definition of α,

$$\|Gz\|^2 \geqslant \|u\|^2 + \|w\|^2 - 2\cos\alpha\, \|u\|\, \|w\|.$$

Since $2ab \leqslant a^2 + b^2$ for all real a, b, we have

$$\|Gz\|^2 \geqslant (1-\cos\alpha)(\|u\|^2 + \|w\|^2).$$

From $1 - \cos\alpha \geqslant (1/2)\sin^2\alpha$ and Lemma 4.1, we arrive at

$$\|Gz\|^2 \geqslant \frac{\theta^2}{2\Lambda^2}(\|u\|^2 + \|w\|^2). \tag{4.6}$$

Theorem 2.4 provides us with an estimate on $\|u\|$:

$$\begin{aligned} \|u\|^2 &= \|PAP^{\perp}z\|^2 + \|P^{\perp}AP^{\perp}z\|^2 \\ &\geqslant \|P^{\perp}AP^{\perp}z\|^2 \\ &\geqslant \theta^2\, \|P^{\perp}z\|^2. \end{aligned}$$

Using this and $w = Pz$ in (4.6) results in

$$\|Gz\|^2 \geqslant \frac{\theta^2}{2\Lambda^2}(\theta^2\, \|P^{\perp}z\|^2 + \|Pz\|^2), \tag{4.7}$$

which yields both the invertibility of G and (4.5) as immediate consequences. ▮

Several remarks are in order:

(1) In the case of thin-plate splines, the invertibility of G was established by Duchon [2], who used methods much different from ours. The invertibility of G in the general case is a result of Madych and Nelson [7] and Micchelli [9].

(2) To obtain true polynomial reproduction, we assumed that the rank of P was the dimension of the appropriate polynomial space. This requires that the underlying data set have a $\pi_{m-1}(\mathbb{R}^s)$ unisolvent subset. If the data set fails to have such a subset, then polynomial reproduction becomes impossible, because some nonzero polynomials of degree $m-1$ or less will vanish on the data. Equivalently, $\operatorname{rank}(P)$ will fall below the dimension of $\pi_{m-1}(\mathbb{R}^s)$.

V. Interpolation without Polynomial Reproduction

We now turn our attention to the interpolation problem that arises from using $F \in \mathscr{RN}^{\infty}_{m,c}$, but that no longer demands polynomial reproducibility. Thus, we wish to investigate the invertibility of the interpolation matrix itself.

For nontrivial $F \in \mathscr{RN}^{\infty}_{1}$, the interpolation matrix A was shown to be invertible, provided $F \neq a + br^2$ and $F(0) \geqslant 0$ [7, 9]. Estimates on $\|A^{-1}\|$ were obtained in [1, 10].

Of course, not all $F \in \mathscr{RN}^{\infty}_{1}$ satisfy $F(0) \geqslant 0$. However, given an F, one may choose a constant c so that $H(r) := F(r) + c$ (which is still in $\mathscr{RN}^{\infty}_{1}$) satisfies $H(0) \geqslant 0$. We will show that something similar happens when $m > 1$. When $F \in \mathscr{RN}^{\infty}_{m,c}$, $m > 1$, we will provide two ways to select a polynomial $p(t)$, with degree $m-1$, such that

$$H(r) := F(r) + p(r^2) \in \mathscr{RN}^{\infty}_{m,c}$$

yields an invertible interpolation matrix. We will also obtain estimates on inverses of the corresponding interpolation matrices. However, if $m > 1$, our choice of $p(r^2)$ does depend on the number of data points.

We begin with two lemmas. In what follows, $\sigma(A)$ will denote the spectrum of A.

Lemma 5.1. *Let A be a self-adjoint matrix and let P_k denote the orthogonal projection onto some k dimensional subspace $V \subset \mathbb{R}^N$. If $\sigma(P_k A P_k) \subset [\varepsilon, \infty)$, $\varepsilon > 0$, and $\sigma(P_k^{\perp} A P_k^{\perp}) \subset (-\infty, -\varepsilon]$, then A has k positive eigenvalues, $N-k$ negative eigenvalues, and $\sigma(A) \subset (-\infty, -\varepsilon] \cup [\varepsilon, \infty)$. In particular,*

$$\|A^{-1}\| \leqslant 1/\varepsilon.$$

Proof. It suffices to show that A has k positive eigenvalues contained in $[\varepsilon, \infty)$, since the same type of argument shows that $B := -A$ has $N-k$ positive eigenvalues contained in $[\varepsilon, \infty)$.

Let σ_1 denote the maximum eigenvalue of A with associated eigenvector ω_1. Then

$$\sigma_1 = \max_{\|x\|=1} \langle Ax, x\rangle \geqslant \max_{\substack{\|y\|=1\\ P_k y = y}} \langle Ay, y\rangle = \max_{\substack{\|y\|=1\\ P_k y = y}} \langle P_k A P_k y, y\rangle \geqslant \varepsilon.$$

For the case $k=1$, we are done. Let $k \geqslant 2$. If $\sigma_2 (\sigma_2 \leqslant \sigma_1)$ denotes the next largest eigenvalue of A, then

$$\sigma_2 = \max_{\substack{\|x\|=1\\ \langle x, \omega_1\rangle = 0}} \langle Ax, x\rangle.$$

Note that $W_1^\perp$ is an $N-1$ dimensional space while the range of P_k is k dimensional. Provided that $N-1+k > N$, there exists a nontrivial y in the intersection of these vector spaces so,

$$\sigma_2 = \max_{\substack{\|x\|=1\\ \langle x, w_1\rangle = 0}} \langle Ax, x\rangle \geqslant \max_{\substack{\|y\|=1\\ \langle y, w_1\rangle = 0\\ P_k y = y}} \langle Ay, y\rangle \geqslant \varepsilon.$$

In this way, one may continue finding positive eigenvalues up to σ_k. ∎

Remarks. (1) In the $m=1$ case, where A is the interpolation matrix associated with $F \in \mathscr{RN}_1^\infty$, one can use Lemma 5.1 to recover Ball's observation [1] that if $F(0) \geqslant 0$, and if

$$\delta := \inf\left\{ -\langle Ac, c\rangle : \sum c_j = 0 \right\} > 0,$$

then $\|A^{-1}\| \leqslant 1/\delta$. To do this, first note that such an F is nonnegative, and so the entries of A are nonnegative, too. Consequently,

$$\delta_+ \equiv \frac{1}{N} (1 \cdots 1) A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} > 0.$$

Taking $V = \operatorname{Span}\{(1 \cdots 1)^T\}$, with P_1 being the projection onto V, we see that $\sigma(P_1 A P_1) \subset [\delta_+, \infty)$. Since by assumption we also have $\sigma(P_1^\perp A P_1^\perp) \subset (-\infty, -\delta]$, it is clear that A has one positive eigenvalue, λ_+, and $N-1$ negative eigenvalues. If we label these $-\lambda_{N-1} \leqslant -\lambda_{N-2} \leqslant \cdots \leqslant -\lambda_1$, then from Lemma 5.1, $\lambda_1 \geqslant \delta$. Also, $\lambda_1 \leqslant \lambda_+$, for Trace $A = \lambda_+ - \lambda_1 - \cdots - \lambda_{N-1} \geqslant 0$. Consequently, $\|A^{-1}\| = 1/\lambda_1 \leqslant 1/\delta$.

(2) In Lemma 5.1, it is clear that if $\sigma(P_k A P_k) \subset [\varepsilon_+, \infty)$ and $\sigma(P_k^\perp A P_k^\perp) \subset (-\infty, -\varepsilon_-]$, with $\varepsilon_+ \neq \varepsilon_-$, but both positive, then $\sigma(A) \subset (-\infty, -\varepsilon_-] \cup [\varepsilon, \infty)$ and $\|A^{-1}\| \leqslant \max\{\varepsilon_+^{-1}, \varepsilon_-^{-1}\}$.

LEMMA 5.2. *Let W be a real vector space with inner product $\langle\ ,\ \rangle$, and let $U \subseteq W$ be a subspace. In addition, let S and T be self-adjoint linear transformations that satisfy these conditions*:

(i) $\langle Sw, w\rangle \geqslant 0$, *for all* $w \in W$.

(ii) $\langle Su, u\rangle \geqslant \sigma \|u\|^2$, $\sigma > 0$, *for all* $u \in U$.

(iii) $\langle Su', u'\rangle = 0$ *and* $\langle Tu', u'\rangle \geqslant \tau \|u'\|^2$, *where* $\tau > 0$, *for all* $u' \in U^{\perp}$.

Then, one may choose γ so that, for all $w \in W$, $\gamma S + T$ satisfies

$$\langle (\gamma S + T) w, w\rangle \geqslant \frac{\tau}{2} \|w\|^2.$$

Proof. Since $W = U \oplus U^{\perp}$, we can write $w \in W$ as $w = \alpha u + \beta u'$, where u, u' are unit vectors in $U, U^{\perp}$, respectively, and $\|w\|^2 = \alpha^2 + \beta^2$. If γ is any real number, then

$$\begin{aligned}\langle (\gamma S + T) w, w\rangle = {}& \gamma\{\alpha^2 \langle Su, u\rangle + 2\alpha\beta \langle Su, u'\rangle + \beta^2 \langle Su', u'\rangle\} \\ & + \alpha^2 \langle Tu, u\rangle + 2\alpha\beta \langle Tu, u'\rangle + \beta^2 \langle Tu', u'\rangle.\end{aligned}$$

Since S is nonnegative in W,

$$|\langle Su, u'\rangle| \leqslant \sqrt{\langle Su, u\rangle}\, \sqrt{\langle Su', u'\rangle},$$

and so, by (iii), $\langle Su, u'\rangle = 0$. Hence,

$$\begin{aligned}\langle (\gamma S + T) w, w\rangle = {}& \alpha^2 \{\gamma \langle Su, u\rangle + \langle Tu, u\rangle\} \\ & + 2\alpha\beta \langle Tu, u'\rangle + \beta^2 \langle Tu', u'\rangle.\end{aligned}$$

Next, by (ii), (iii), and $\|u\| = \|u'\| = 1$,

$$\langle (\gamma S + T) w, w\rangle \geqslant \alpha^2(\gamma\sigma - \|T\|) - 2\,|\alpha|\,|\beta|\,\langle Tu, u'\rangle + \beta^2 \tau.$$

Since $\|u\| = \|u'\| = 1$, $|\langle Tu, u'\rangle| \leqslant \|T\|$. This and the inequality $2ab \leqslant \varepsilon^{-1} a^2 + \varepsilon b^2$ imply that

$$\langle (\gamma S + T) w, w\rangle \geqslant \alpha^2 \left(\gamma\sigma - \|T\| - \frac{\|T\|}{\varepsilon}\right) + \beta^2 (\tau - \varepsilon \|T\|).$$

Choose $\varepsilon = \tau/(2\,\|T\|)$ to get

$$\langle (\gamma S + T) w, w\rangle \geqslant \alpha^2 \left(\sigma\gamma - \|T\| - \frac{2\,\|T\|^2}{\tau}\right) + \beta^2 \left(\frac{\tau}{2}\right).$$

Finally, pick $\gamma = (1/\sigma)[\|T\| + 2\tau^{-1} \|T\|^2 + \tau/2]$ to complete the proof. ∎

We are now ready to prove the main results of this section. Both theorems we are about to state concern the invertibility of the interpolation matrix coming from an $F \in \mathscr{RN}_{m,c}^{\infty}$ that has been modified by adding a polynomial in r^2. One of the theorems is the direct analog of results proved in the case $m=1$, while the other shows that F may be modified so that the resulting interpolation matrix will be negative definite.

THEOREM 5.3. *Let $F \in \mathscr{RN}_{m,c}$ be such that $(d^m/dt^m)[F(\sqrt{t})]$ is non-constant. Then there exist scalars $a_1, a_2, ..., a_m$ such that*

$$H(r) := F(r) + a_m + a_{m-1} r^2 + \cdots + a_1 r^{2m-2} \tag{5.1}$$

is in $\mathscr{RN}_{m,c}^{\infty}$ and the interpolation matrix B corresponding to H is invertible, has μ positive eigenvalues, $N-\mu$ negative eigenvalues, and satisfies

$$\|B^{-1}\| \leqslant \theta^{-1}, \tag{5.2}$$

where θ is as in Theorem 2.4 and $\mu = \dim V_{m-1}$, where V_{m-1} is defined in (2.2).

Proof. Note that $H \in \mathscr{RN}_{m,c}^{\infty}$, since $(d^m/dr^m)(H(\sqrt{r}) - F(\sqrt{r})) = 0$. As before, let

$$V_k := \{v \in \mathbb{R}^N : v_j = p(x_j),\ p \in \pi_k(\mathbb{R}^s)\}.$$

It is easy to see that V_{m-1} can be decomposed in the following way:

$$V_{m-1} = (V_{m-2}^{\perp} \ominus V_{m-1}^{\perp}) \oplus (V_{m-3}^{\perp} \ominus V_{m-2}^{\perp}) \oplus \cdots \oplus V_0.$$

The proof proceeds by induction. First, let $W_1 = V_{m-2}^{\perp} \ominus V_{m-1}^{\perp} = U_1$, so relative to W_1, $U_1^{\perp} = \{0\}$. Let S_1 be the matrix of $(-1)^{m-1} r^{2m-2}$, and let T be the interpolation matrix of F. By Lemma 1.4,

$$\langle S_1 w, w \rangle \geqslant \sigma_1 \|w\|^2, \qquad w \in W_1.$$

Since $U_1^{\perp} = \{0\}$, the τ_1 that appears in Lemma 5.2 is arbitrary, so choose it to be $\tau = \tau_1 = 2^m \theta$. Thus, we can find γ_1 so that

$$\langle (\gamma_1 S_1 + T_1) w, w \rangle \geqslant 2^{m-1} \theta \|w\|^2, \qquad w \in W_1.$$

Choose $a_1 = (-1)^{m-1} \gamma_1$.

Second, define the following:

$$\begin{cases} W_2 = (V_{m-3}^{\perp} \ominus V_{m-2}^{\perp}) \oplus (V_{m-2}^{\perp} \ominus V_{m-1}^{\perp}) \\ U_2 = V_{m-3}^{\perp} \ominus V_{m-2}^{\perp} \\ S_2 = \text{interpolation matrix of } (-1)^{m-2} r^{2m-4} \\ T_2 = \gamma_1 S_1 + T_1. \end{cases}$$

Note that relative to W_2, $U_2^{\perp} = W_1$. Consequently, the inequality above implies that on $U_2^{\perp}$, T_2 is bounded below by $\tau_2 = 2^{m-1}\theta$. From Lemma 1.4, it is clear that $\langle S_2 u_2', u_2' \rangle = 0$, $u_2' \in U_2^{\perp} = W_1$, and that there is a constant σ_2 such that

$$\langle S_2 u_2, u_2 \rangle \geqslant \sigma_2 \|u_2\|^2, \qquad u_2 \in U_2.$$

Applyig Lemma 5.2 again gives us a constant γ_2 such that

$$\langle (\gamma_2 S_2 + T_2) w, w \rangle \geqslant 2^{m-2}\theta \|w\|^2, \qquad w \in W_2.$$

Choose $a_2 = (-1)^{m-2}\gamma_2$.

We may proceed in this way until all the coefficients have been determined. The result is then a function $H \in \mathscr{R}\mathscr{N}_{m,c}^{\infty}$ with the property that its interpolation matrix B satisfies

$$\langle Bw, w \rangle \geqslant \theta \|w\|^2, \qquad w \in V_{m-1}.$$

On the other hand, if we let A be the interpolation matrix associated with F, then Lemma 1.4 implies

$$\langle Bw, w \rangle = \langle Aw, w \rangle, \qquad w \in V_{m-1}^{\perp}.$$

From Theorem 2.4, we then obtain

$$\langle Bw, w \rangle \leqslant -\theta \|w\|^2, \qquad w \in V_{m-1}^{\perp}.$$

Applying Lemma 5.1 then yields the theorem. ∎

THEOREM 5.4. *Let $F \in \mathscr{R}\mathscr{N}_{m,c}^{\infty}$ be such that $(d^m/dt^m)[F(\sqrt{t})]$ is non-constant. Then there exist scalars $b_1, ..., b_m$ such that*

$$K(r) := F(r) + b_m + b_{m-1} r^2 + \cdots + b_1 r^{2m-2} \tag{5.3}$$

is in $\mathscr{R}\mathscr{N}_{m,c}^{\infty}$ and the interpolation matrix C corresponding to K is invertible, negative definite, and satisfies

$$\|C^{-1}\| \leqslant \frac{2^m}{\theta}, \tag{5.4}$$

where θ is as in Theorem 2.4.

Proof. The proof again follows by induction. Start by observing that if

$$W_1 = (V_{m-2}^{\perp} \ominus V_{m-1}^{\perp}) \oplus V_{m-1}^{\perp},$$

then, with $U_1 = V_{m-2}^{\perp} \ominus V_{m-1}^{\perp}$ and $U_1^{\perp}$ its orthogonal complement in W_1 (i.e., $U_1^{\perp} = V_{m-1}^{\perp}$), the matrices

$$\begin{cases} S_1 := \text{interpolation matrix of } (-1)^{m-1} r^{2m-2} \\ T_1 := -A \end{cases}$$

will satisfy the conditions of Lemma 5.2, provided

$$\tau_1 = \theta \qquad \text{and} \qquad \sigma_1 = \inf\{\langle S_1 v, v\rangle : v \in V_{m-2}^{\perp} \ominus V_{m-1}^{\perp}\}.$$

That this is so is a consequence of Lemma 1.4 and Theorem 2.4. Lemma 5.2 then implies that there is a γ_1 such that

$$\langle (\gamma_1 S_1 - T_1) w, w\rangle \geqslant \frac{\theta}{2} \|w\|^2, \qquad w \in W_1.$$

Choose $b_1 = -(-1)^{m-1}\gamma_1$.

In the next step, we work with

$$\begin{cases} W_2 := (V_{m-3}^{\perp} \ominus V_{m-2}^{\perp}) \oplus V_{m-2}^{\perp} \\ S_2 := \text{interpolation matrix of } (-1)^{m-2} r^{2m-4} \\ T_2 := \gamma_1 S_1 - A. \end{cases}$$

After noting that $W_1 = V_{m-2}^{\perp}$, the argument is virtually identical to that above. The inequality analogous to the one above is

$$\langle (\gamma_2 S_2 + T_2) w, w\rangle \geqslant \frac{\theta}{2^2} \|w\|^2, \qquad w \in W_2.$$

Choose $b_2 = (-1)^{m-2}\gamma_2$.

Continuing in this way, one may choose the b's in (5.3). The interpolation matrix C obviously satisfies

$$\langle Cw, w\rangle \leqslant -\frac{\theta}{2^m} \|w\|^2, \qquad w \in \mathbb{R}^N.$$

Consequently, C is negative definite and invertible, and $\|C^{-1}\|$ satisfies (5.4). ∎

Remark. As far as we know, no one has interpolated using the functions H and K constructed above. We would certainly be interested in numerical tests that used these functions to interpolate scattered data. We close by pointing out that although norm estimates in B^{-1} are better than on C^{-1}, in the sense of being smaller, C^{-1} can be computed using steepest descent methods—because C is negative definite.

VI. THIN-PLATE SPLINES

Thin-plate splines are radial functions that are in $\mathscr{RN}^{\infty}_{m,c}$ and that minimize certain Sobolev norms. Because these functions have been extensively employed in interpolation problems, we have chosen them to illustrate our results.

Let $d=1, 2, 3, ...$, and let $m>d/2$, where m is an integer. Define the functions

$$g_{m,d}(\sigma):=\begin{cases}\dfrac{(-1)^{m-[d/2]}\pi\sigma^{m-d/2}}{\Gamma(m+1-d/2)}, & d \text{ odd}\\[2ex] \dfrac{(-1)^{m+1-d/2}\sigma^{m-d/2}\ln\sigma}{\Gamma(m+1-d/2)}, & d \text{ even.}\end{cases}$$

The thin-plate spline associated with the pair m, d is

$$F_{m,d}(r):=-g_{m,d}(r^2).$$

Using induction, one may verify that the mth derivative of $g_{m,d}$ is given by

$$g^{(m)}_{m,d}(\sigma)=\frac{(-1)^m\Gamma(d/2)}{\sigma^{d/2}}.$$

A standard Laplace transform formula then yields

$$(-1)^m g^{(m)}_{m,d}(\sigma)=\int_0^{\infty} t^{d/2-1}e^{-\sigma t}\,dt,$$

which of course shows that $F_{m,d}\in\mathscr{RN}^{\infty}_{m,c}$ and implies that the measure $d\eta(t)$ in Theorem 2.4 is given by

$$d\eta(t)=t^{d/2-1}\,dt.$$

Using this measure allows us to calculate the quantity θ that appears in Theorem 2.4. We have

$$\theta=C_s q^{-s}\int_0^{\infty} t^{(d-s)/2-m-1}e^{-\delta^2q^{-2}t^{-1}}\,dt.$$

If we substitute $u=\delta^2q^{-2}t^{-1}$ above, the integral that results is standard. Doing it yields

$$\theta=C_s q^{2m-d}\delta^{d-s-2m}\Gamma\left(m+\frac{s-d}{2}\right), \tag{6.1}$$

where C_s and δ are given by

$$\delta := 12\left(\frac{\pi\Gamma^2((s+2)/2)}{9}\right)^{1/(s+1)} \quad \text{and} \quad C_s := \frac{\delta^2}{2^{s+1}\Gamma((s+2)/2)}. \tag{6.2}$$

The first example that we wish to look at is the case in which $m=1$ and $d=1$, and the CNDR function is $F_{1,1}(r)=2\pi^{1/2}r$. We chose this case because, when $s=1$, the best estimate on $\|A^{-1}\|$is known [1]. If a constant factor is accounted for, the result from [1] is that $\|A^{-1}\| \leqslant \pi^{-1/2}q^{-1}$, which is sharp. To get an estimate using our results, first note that with $m=d=s=1$, we have that $\delta=2\pi$ and that $C_1=2\pi^{3/2}$. From (6.1), it follows that $\theta=q/2\sqrt{\pi}$. Applying Corollary 2.6 then yields the estimate

$$\|A^{-1}\| \leqslant 2\sqrt{\pi}\, q^{-1}.$$

Thus our results yield an estimate that is a factor of 2π larger than the best possible estimate, but that has the same q dependence.

The two most important cases occur when $m=2$ and $s=d=2$ or $s=3$, $d=1$. In these cases, we have the following θ-values:

$$\theta=\begin{cases} 1.751\times 10^{-3}q^2, & \text{when} \quad s=d=m=2; \\ 7.351\times 10^{-6}q^3, & \text{when} \quad m=2 \quad \text{and} \quad s=3, \quad d=1. \end{cases} \tag{6.3}$$

We dealt with three interpolation matrices: G defined in Theorem 4.2; B, defined in Theorem 5.3; and C, defined in Theorem 5.4. For the function $F_{2,2}=-r^2\ln r^2$ and $F_{2,1}=(4\pi^{1/2}/3)r^3$, only (6.3) is required to estimate $\|B^{-1}\|$ and $\|C^{-1}\|$. To estimate $\|G^{-1}\|$, we also need

$$\|AP^{\perp}\| \leqslant \|A\| \leqslant \left(\sum A_{j,k}^2\right)^{1/2} \leqslant N\max|F(\|x_j-x_k\|)|. \tag{6.4}$$

Combining the theorems mentioned above with (6.3) and (6.4) results in the table below:

	$\|G^{-1}\|$	$\|B^{-1}\|$	$\|C^{-1}\|$
$-r^2\ln r^2$	$(808D^2N\ln D)q^{-2}\max\{1, 571q^{-2}\}$	$571q^{-2}$	$2,284q^{-2}$
$4\sqrt{\pi}\,r^3/3$	$(4.54\times 10^5 q^{-3}ND^3)\max\{1, q^{-3}\}$	$1.36\times 10^5 q^{-3}$	$5.44\times 10^5 q^{-3}$

In the table above, D, which is the diameter of the data set, was assumed to be larger than 2. Also, although N, D, and q are independent, one can show (see [10, Sect. 7]) that

$$N \leqslant \left(\frac{D+2q}{2q}\right)^2.$$

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